hep–th/9707198
IASSNS–HEP–97/86

Target Space Duality for (0,2) Compactifications

Ralph Blumenhagen[1]

[1] *School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton NJ 08540, USA*

Abstract

The moduli spaces of two $(0,2)$ compactifications of the heterotic string can share the same Landau-Ginzburg model even though at large radius they look completely different. It was argued that such a pair of $(0,2)$ models might be connected via a perturbative transition at the Landau-Ginzburg point. Situations of this kind are studied for some explicit models. By calculating the exact dimensions of the generic moduli spaces at large radius, strong indications are found in favor of a different scenario. The two moduli spaces are isomorphic and complex, Kähler and bundle moduli get exchanged.

07/97

[1] e–mail: blumenha@sns.ias.edu

1. Introduction

String compactifications to four dimensions with $N = 1$ supersymmetry are the most promising class of models to describe our physical universe. For the heterotic string the condition of $N = 1$ supersymmetry leads one to models with $(0, 2)$ supersymmetry from the world sheet perspective [1]. This class of models has been the subject of study all over the last decade [2–10]. However, especially during recent years not only the left-right symmetric subclass of $(2, 2)$ supersymmetric models was under investigation but also new results were obtained establishing more general $(0, 2)$ supersymmetric models. Former doubts about the consistency of such models could be shown to be not justified at least for the class of $(0, 2)$ models described by linear σ-models [5]. Furthermore, exactly solvable $(0, 2)$ models were constructed and identified with some special points in the moduli space of such linear σ-models, showing directly that these models are consistent [6].

For the class of $(2, 2)$ models the target space duality commonly known as mirror symmetry turned out to be of primary importance to solve the model on the Kähler moduli space [11–13]. These results were used later to establish a new non-perturbative duality between models with $N = 2$ supersymmetry, namely between the type II A string on a $K3$ fibered Calabi-Yau threefold and the heterotic string on $K3 \times T_2$ [14]. The local analysis of this stringy duality reproduced the results of Seiberg/Witten on the Coloumb moduli space of $N = 2$ gauge theories [15]. In [10] the notion of mirror symmetry was generalized to $(0, 2)$ models, by showing that some of the mirror constructions known from the $(2, 2)$ context, like orbifolding, can be carried over to the $(0, 2)$ case.

F-theory on elliptically fibered Calabi-Yau fourfolds provides another huge class of consistent $N = 1$ supersymmetric models in four dimensions [16]. These models are defined as type II B compactifications on so-called D-manifolds, which are not necessarily Ricci-flat and support D-branes on some submanifolds. If the fourfold is also $K3$ fibered then such a compactification is expected to be dual to an elliptically fibered heterotic $(0, 2)$ model with in general a number of five-branes wrapped around the toroidal fiber.

The subject of this paper is another duality holding in the class of $N = 1$ models. It was first observed in [4] and further elaborated in [8] that two at first sight different $(0, 2)$ models can share the same Landau-Ginzburg model at small radius. Since in general the number of gauge singlets at the Landau-Ginzburg point was known to be bigger than in the large radius geometric phase one could imagine a situation similar to a conifold transition [17]. One starts at large radius on one $(0, 2)$ model, moves down to small radius, hits the

Landau-Ginzburg point, turns on one of the new moduli and finally finds oneself on the other $(0,2)$ model. The aim of this paper is to argue that such a transition is very unlikely to happen and can be excluded definitely in one of our examples involving the well known quintic. Technically, what will be done is to calculate the dimension of the three relevant cohomology classes $H^1(M,T)$, $H^2(M,T)$ and $H^1(M,\mathrm{End}(V))$ giving complex, Kähler and bundle moduli of the $(0,2)$ compactification, respectively. The result is that in all studied examples the dimension of the large radius geometric moduli space for two dual models agrees. This is so, even if the Landau-Ginzburg model provides additional singlets. This result points not to a transition to occur at the Landau-Ginzburg point, but instead to an actual target space duality between different $(0,2)$ moduli spaces, where complex, Kähler and bundle moduli get exchanged. The possibility of an isomorphism of moduli spaces was mentioned before in [8], where a third scenario was also taken into account. Two models could be related by something similar to a flop transition, thus describing an overall model in different regions of its moduli space. Since the whole bundle moduli space of the quintic is visible in the parameters of the linear σ-model, it can be compactified. Consequently, one does not have any boundaries, so that the possibility of a flop like transition will be excluded in the following discussion.

This paper is organized as follows. In section two some facts about linear σ-models and the duality at the Landau-Ginzburg point are briefly reviewed. In section three one example is discussed in very much detail. It deals with the quintic and a dual $(0,2)$ candidate, for which the bundle valued cohomology classes are calculated. The technical aspects are discussed fairly explicitly, not to bother the reader with some boring technical aspects but to provide some further compressed reference for such calculations. The available techniques are scattered around the literature and are most often limited to ordinary projective spaces [18]. Whereas, here one has to deal with more general toric varieties. In section four the same techniques are applied to some further examples involving the sextic $\mathbb{P}_{1,1,1,1,2}[6]$, for which two dual $(0,2)$ models are found featuring the same dimension of the geometric moduli spaces.

2. Linear σ-models

The primary reason $(0,2)$ models have become accessible to study in recent times is the development of the gauged linear σ-model by Witten [19]. This model is a relatively tractable massive two-dimensional field theory which is believed, under suitable conditions,

to flow in the infrared to a non-trivial superconformal field theory. One of the more interesting features of the linear σ-model is its various connected vacua, or phases. At low energies, these phases appear to correspond to theories such as a non-linear σ-model, a Landau-Ginzburg orbifold, or some other more peculiar theory like a hybrid model. The linear σ-model provides a natural setting in which the relation between some of these various types of theories can be studied.

Let us begin by describing the fields in the $(0,2)$ linear σ-model. To shorten the notation it is assumed in the following review that there is only one $U(1)$ gauge field. The generalization to more gauge fields is straightforward and can be found in [7,19]. There are two sets of chiral superfields: $\{X_i | i = 1, \ldots, N_x\}$ with $U(1)$ charges ω_i and $\{P_l | l = 1, \ldots, N_p\}$ with $U(1)$ charges $-m_l$. Furthermore, there are two sets of Fermi superfields: $\{\Lambda^a | a = 1, \ldots, N_\Lambda\}$ with charges n_a and $\{\Gamma^j | j = 1, \ldots, N_\Gamma\}$ with charges $-d_j$. The superpotential of the linear σ-model is given by,

$$S = \int d^2 z d\theta \left[\Gamma^j W_j(X_i) + P_l \Lambda^a F_a^l(X_i) \right], \tag{2.1}$$

where G_j and F_a^l are quasihomogeneous polynomials whose degree is fixed by requiring charge neutrality of the action. To ensure the absence of gauge anomalies the following conditions have to be satisfied:

$$\begin{aligned} &\sum \omega_i = \sum d_j, \\ &\sum n_a = \sum m_l, \\ &\sum d_j^2 - \sum w_i^2 = \sum m_l^2 - \sum n_a^2. \end{aligned} \tag{2.2}$$

If there is more than one $U(1)$ gauge field the linear conditions have to be satisfied for every single $U(1)$ and the quadratic condition for every pair of $U(1)$s. Thus, N different $U(1)$ symmetries give rise to $N(N+1)/2$ quadratic conditions.

In the large radius limit $r \gg 0$, the model describes a $(0,2)$ non-linear σ-model on a generally singular weighted projective space, $\mathbb{P}_{\omega_1, \ldots, \omega_{N_x}}[d_1, \ldots, d_{N_\Gamma}]$, with a coherent sheaf of rank $N_\Lambda - N_p - N_F$ defined as the cohomology of the monad

$$0 \to \bigoplus_{i=1}^{N_F} \mathcal{O} \overset{\otimes E_a^i}{\to} \bigoplus_{a=1}^{N_\Lambda} \mathcal{O}(n_a) \overset{\otimes F_a}{\to} \bigoplus_{l=1}^{N_p} \mathcal{O}(m_l) \to 0. \tag{2.3}$$

Here N_F additional fermionic gauge symmetries have been introduced to make the model consistent. For an extended discussion of these fermionic gauge symmetries take a look into [8]. In the subsequent sections the notation

$$V(n_1, \ldots, n_{N_\Lambda}; m_1, \ldots, m_{N_p}) \rightarrow \mathbb{P}_{\omega_1,\ldots,\omega_{N_x}}[d_1, \ldots, d_{N_\Gamma}] \tag{2.4}$$

will be used for the singular configuration. For the situation where $N_p = 1$ and $r \ll 0$, the low-energy physics is described by a Landau-Ginzburg orbifold with a superpotential

$$W(X_i, \Lambda^a, \Gamma^j) = \sum_j \Gamma^j W_j(X_i) + \sum_a \Lambda^a F_a(X_i). \tag{2.5}$$

It was first observed in [4] that in this superpotential the constraints G_j and F_a appear on equal footing, so that in particular an exchange of them does not change the Landau-Ginzburg model as long as all anomaly cancellation conditions are satisfied. In [8] this duality was further elaborated showing that this exchange is still possible after resolving the generically singular base manifold. The fact that at the Landau-Ginzburg point the number of gauge singlets is usually bigger than in the geometric large radius limit was viewed as a hint that a transition from one $(0,2)$ model to another one takes place right at the Landau-Ginzburg point. This transition would be a perturbative analogue of the non-perturbative conifold transition for type II models [17]. It will be shown in the next section that at least for some simple examples such a scenario does not survive more refined tests.

3. The quintic and its (0,2) dual

The quintic is probably the most studied example of a Calabi-Yau compactification in the literature. It is given by a quintic hypersurface M in the projective space $\mathbb{P}_4$. The bundle V_M over $\mathbb{P}_4[5]$ is a deformation of the tangent bundle and can be described as the cohomology of the monad

$$0 \rightarrow \mathcal{O}|_M \rightarrow \bigoplus_{a=1}^{5} \mathcal{O}(1)|_M \rightarrow \mathcal{O}(5)|_M \rightarrow 0. \tag{3.1}$$

The notation $\mathcal{O}(n)|_M$ means the line bundle on the ambient space with first Chern class $n\eta$ restricted to the hypersurface M. What is needed for the following is the dimension of the moduli space. At large radius one finds $h^1(M, V_M) = 101$ complex deformations,

$h^2(M, V_M) = 1$ Kähler deformations and $h^1(M, \mathrm{End}(V_M)) = 224$ bundle deformations adding up to a total dimension of 326. In this particular case one finds the same number of moduli at the Landau-Ginzburg point [20]. Moreover, E. Silverstein and E. Witten [5] have shown that there is no perturbative superpotential generated for these moduli, thus one has a nice compact 326 complex dimensional moduli space of vacua, containing singular loci of at least complex codimension one.

Is there another $(0, 2)$ model which agrees with the quintic at the Landau-Ginzburg point? Indeed there is, the singular configuration can be written as

$$V(1, 1, 1, 2; 5) \to \mathbb{P}_{1,1,1,1,1,3}[4, 4]. \tag{3.2}$$

One new coordinate of weight $\omega = 3$ was introduced, which however generates a mass term $\lambda_4 x_6$ in the $(0, 2)$ superpotential and therefore can be integrated out. We already know that there do not exist any further moduli at the Landau-Ginzburg point which are not moduli of the quintic. Therefore, there cannot be a transition from the quintic to this dual model. Every deformation around the quintic Landau-Ginzburg point corresponds to a deformation of the dual model. Thus, the most natural scenario is, that the two moduli spaces of the quintic and the dual model are isomorphic. One necessary condition for this picture to be true is that the dual model at large radius also has a 326 dimensional moduli space. Thus, one is facing the task of calculating bundle valued cohomology for $(0, 2)$ models. This is a quite technical process but I am under the impression that these techniques [18], in particular those for calculating $h^1(M, \mathrm{End}(V_M))$, are not so well known and therefore, they will be reviewed and generalized here.

3.1. Toric resolution

To begin with, since the base manifold of the model

$$V(1, 1, 1, 2; 5) \to \mathbb{P}_{1,1,1,1,1,3}[4, 4] \tag{3.3}$$

contains a $\mathbb{Z}_3$ singularity one has to resolve the ambient space. To this end, methods known from toric geometry are used [13][1]. The vertices of the fan describing the resolved toric variety are

$$\begin{aligned} &v_1 = (1, 0, 0, 0, 0),\ v_2 = (0, 1, 0, 0, 0),\ v_3 = (0, 0, 1, 0, 0),\ v_4 = (0, 0, 0, 1, 0), \\ &v_5 = (0, 0, 0, 0, 1),\ v_6 = (-1, 0, 0, 0, 0),\ v_7 = (-3, -1, -1, -1, -1). \end{aligned} \tag{3.4}$$

[1] For some of the calculations involving toric varieties the maple packages *Schubert* and *Puntos* have been used [21]

The only star subdivision is

$$\begin{aligned}\text{Cones} = \{&[1,2,3,4,5],[1,2,3,4,7],[1,2,3,5,7],[1,2,4,5,7],[1,3,4,5,7],\\ &[2,3,4,5,6],[2,3,4,6,7],[2,3,5,6,7],[2,4,5,6,7],[3,4,5,6,7]\},\end{aligned} \tag{3.5}$$

which yields the Stanley-Reisner ideal

$$\mathrm{SR} = \{x_1\, x_6,\ \ x_2\, x_3\, x_4\, x_5\, x_7\}. \tag{3.6}$$

The charges of the fields in the linear σ-model are given by the kernel of the matrix of all vertices. In our case there are two $U(1)$s under which the fields carry charges. For the fields defining the base threefold these charges are presented in Table 3.1.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	Γ_1	Γ_2
1	0	0	0	0	1	0	-1	-1
3	1	1	1	1	0	1	-4	-4

Table 3.1: *Charges for the base.*

The resolution of the sheaf has to be done in such a way that all quadratic anomaly cancellation conditions for the chiral fermions are satisfied and that the sheaf agrees with the unresolved one on the singular locus. In our case the resolution is given by assigning the $U(1)$ charges in Table 3.2 to the left moving fermions in the linear σ-model.

λ_1	λ_2	λ_3	λ_4	λ_5	p
1	0	0	0	0	-1
0	1	1	1	2	-5

Table 3.2: *Charges for the bundle.*

In order to get a sheaf of rank three on the resolved space one has to introduce one fermionic gauge symmetry so that the sheaf is given by the cohomology of the monad

$$0 \to\ \mathcal{O}|_M \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^3 \oplus \mathcal{O}(0,2)|_M \to \mathcal{O}(1,5)|_M \to 0. \tag{3.7}$$

Denoting the two independent divisors in the ambient space A spanning $H_4(A) = H^2(A)$ as η_1 and η_2, a field with charges $(Q_1, Q_2) = (m, n)$ can be regarded as a section of the line bundle $\mathcal{O}(m\eta_1 + n\eta_2)$. These line bundles are written as $\mathcal{O}(m, n)$. In order for the sheaf V_M to be non-singular one has to choose the maps $E_a(x)$ and $F_a(x)$ in such a way

that each set does not vanish simultaneously on M. The general form of the hypersurfaces and functions F_a is

$$\begin{aligned}
&G^{(1,2)}_{(1,4)} = x_1\, P_1(y) + x_6\, P_4(y) \\
&F^1_{(0,5)} = P_5(y), \quad F^{(2,3,4)}_{(1,4)} = x_1\, P_1(y) + x_6\, P_4(y), \quad F^5_{(1,3)} = x_1 + x_6\, P_3(y).
\end{aligned} \tag{3.8}$$

For generic choice of the polynomials $P_n(y)$ with $y = (x_2, x_3, x_4, x_5, x_7)$ vanishing of the G^js and F_as implies either $x_1 = x_6 = 0$ or $x_2 = x_3 = x_4 = x_5 = x_7 = 0$. However, due to the Stanley-Reisner ideal both sets are excluded from the ambient toric variety. For the E_as one can take

$$E^1_{(1,0)} = x_6, \quad E^{(2,3,4)}_{(0,1)} = P_1(y), \quad E^5_{(0,2)} = P_2(y). \tag{3.9}$$

For this choice of data the coherent sheaf defined by the monad (3.7) is actually a vector bundle and we will call it this way in the following discussion. Using the Stanley-Reisner ideal one first calculates the intersection ring of the ambient toric variety

$$81\eta_1^5 - 27\eta_1^4\eta_2 + 9\eta_1^3\eta_2^2 - 3\eta_1^2\eta_2^3 + \eta_1\eta_2^4 \tag{3.10}$$

and afterwards the intersection ring of the complete intersection threefold

$$9\eta_1^3 - 3\eta_1^2\eta_2 + \eta_1\eta_2^2 + 5\eta_2^3. \tag{3.11}$$

Splitting the monad into two exact sequences

$$\begin{aligned}
&(E):\ 0 \to \mathcal{O}|_M \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^3 \oplus \mathcal{O}(0,2)|_M \to \mathcal{E}_M \to 0 \\
&(V):\ 0 \to\ V_M \to \mathcal{E}_M \to \mathcal{O}(1,5)|_M \to 0,
\end{aligned} \tag{3.12}$$

it is straightforward to compute the third Chern class of the bundle V_M. One finds $c_3(V_M) = -200$ which nicely agrees with the Landau-Ginzburg calculation and the result for the quintic. The same computation can be carried out for the tangent bundle leading to the Euler number of the base manifold $\chi(M) = -168$. This is very encouraging and more refined topological numbers will be calculated in the following two subsections. But before that, the phase structure of the Kähler moduli space of the resolved model is briefly discussed in order to ensure that there still is a Landau-Ginzburg phase. The D-terms in the linear σ-model are

$$\begin{aligned}
D_1 &= |x_1|^2 + |x_6|^2 - |p|^2 - r_1 \\
D_2 &= 3|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_7|^2 - 5|p|^2 - r_2
\end{aligned} \tag{3.13}$$

leading to the following four phases in the Kähler moduli space.

Phase I: $r_1 > 0$ and $r_2 - 3r_1 > 0$

This is the Calabi-Yau phase. The x_i take values in the complete intersection in the toric variety and the left moving fermions live on the bundle described by the monad (3.7).

Phase II: $r_2 > 0$ and $r_2 - 3r_1 < 0$

In this second geometric region one has an orbifold phase. The base manifold is the singular $\mathbb{P}_{1,1,1,1,1,3}[4,4]$ with a V bundle over it.

Phase III: $5r_1 - r_2 > 0$ and $r_2 < 0$

This is a Landau-Ginzburg phase, where the expectation values of $|p|^2$ and $|x_6|^2$ are fixed at $-r_2/5$ and $(5r_1 - r_2)/5$, respectively. The remaining coordinate fields have vanishing expectation value and their fluctuations are governed by a superpotential which after integrating out x_1 looks the same as that for the quintic.

Phase IV: $5r_1 - r_2 < 0$ and $r_1 < 0$

This is some hybrid phase, in which $x_1 = x_6 = 0$, $|p|^2 = -r_1$ and the remaining five x_i take values in a quintic $F_1(y) = 0$ with Kähler class $r_2 - 5r_1$. However, since $x_1 = x_6 = 0$ the left and right moving fermions do not take values in the tangent bundle of this quintic, so that one does not simply get a non-linear σ-model. In this phase the $(0,2)$ model seem to memorize its connection to the quintic, as well.

3.2. Complex and Kähler deformations

In general a short exact sequence of sheaves

$$0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0 \tag{3.14}$$

implies a long exact sequence in cohomology

$$0 \to H^0(M,A) \stackrel{\alpha}{\to} H^0(M,B) \stackrel{\beta}{\to} H^0(M,C) \stackrel{\phi}{\to} H^1(M,A) \stackrel{\alpha}{\to} H^1(M,B) \to \ldots . \tag{3.15}$$

The maps α and β in (3.15) are induced from the sheaf homomorphisms in (3.14). For the definition of ϕ it is refered to the mathematical literature [18], but it is emphasized that the definition of ϕ relies on the shortness of the sequence (3.14). In order to use the long exact cohomological sequences implied by (3.12), one has to know the cohomology classes of line bundles restricted to the complete intersection locus. To this end one uses

the Koszul sequence for a complete intersection of K hypersurfaces $\xi = (f_1, \dots, f_K)$ with f_i a section of the line bundle $\mathcal{E}_{f_i}$ over the ambient space

$$0 \to \wedge^K \mathcal{E}^* \xrightarrow{\xi\cdot} \dots \xrightarrow{\xi\cdot} \wedge^2 \mathcal{E}^* \xrightarrow{\xi\cdot} \mathcal{E}^* \xrightarrow{\xi\cdot} \mathcal{O} \xrightarrow{\rho\cdot} \mathcal{O}|_M \to 0. \tag{3.16}$$

Here $\mathcal{E} = \bigoplus \mathcal{E}_{f_i}$ and $\mathcal{O}$ denotes the structure sheaf of the ambient space. In our case, after multiplication of (3.16) with a vector bundle T, one simply obtains the exact sequence

$$0 \to T \otimes \mathcal{O}(-2,-8) \to T \otimes (\mathcal{O}(-1,-4) \oplus \mathcal{O}(-1,-4)) \to T \to T|_M \to 0. \tag{3.17}$$

Throughout the following computation T will always be a line bundle, which means that all one has to know as input are the cohomology classes of line bundles over the ambient toric variety. Luckily, by using the algorithm of [7] one can derive closed formulas for the dimension of these classes. Setting the binomial coefficient of a negative number over a positive number to zero, the only non-zero classes are

$$\begin{aligned}
&h^0(\mathcal{O}(m,n)) = \sum_{l=0}^{m} \binom{n-3l+4}{4}, \quad \text{for } m,n \geq 0 \\
&\text{Čech representative :} \quad P(x_1, x_6, y_i) \\[2ex]
&h^1(\mathcal{O}(-2-m, n-3(m+1))) = \sum_{l=0}^{m} \binom{n-3l+4}{4}, \quad \text{for } m,n \geq 0 \\
&\text{Čech representative on } \{x_1 \neq 0\} \cap \{x_6 \neq 0\} : \quad \frac{P(y_i)}{x_1 x_6 Q(x_1, x_6)} \\[2ex]
&h^4(\mathcal{O}(m, 3m-n-5)) = \sum_{l=0}^{m} \binom{n-3l+4}{4}, \quad \text{for } m,n \geq 0 \\
&\text{Čech representative on } \textstyle\bigcap_i \{y_i \neq 0\} : \quad \frac{P(x_1, x_6)}{y_2 y_3 y_4 y_5 y_7 Q(y_i)} \\[2ex]
&h^5(\mathcal{O}(-2-m, -8-n)) = \sum_{l=0}^{m} \binom{n-3l+4}{4}, \quad \text{for } m,n \geq 0 \\
&\text{Čech representative on } \textstyle\bigcap_i \{y_i \neq 0\} \cap \{x_1 \neq 0\} \cap \{x_6 \neq 0\} : \\
&\quad \frac{1}{x_1 x_6 y_2 y_3 y_4 y_5 y_7 Q(x_1, x_6, y_i)}.
\end{aligned} \tag{3.18}$$

It is checked in many examples that these numbers are consistent with the Euler characteristic $\chi(A, \mathcal{O}(m,n))$ of a line bundle over the ambient space as determined by the Riemann-Roch-Hirzebruch theorem

$$\chi(A, \mathcal{O}(m,n)) = \sum_{q=0}^{\dim A} (-)^q h^q(A, \mathcal{O}(m,n)) = \int_A \left[e^{\lambda} \prod_{i=1}^{N_x} \frac{l_i}{1-e^{-l_i}} \right]. \tag{3.19}$$

In (3.19) λ is the first Chern class of the line bundle $\mathcal{O}(m,n)$ and the l_i denote the first Chern classes of the homogeneous coordinates defining the toric variety. Plugging in all the data from Table 3.1, one obtains for the Euler characteristic

$$\begin{aligned}\chi(A, \mathcal{O}(m,n)) = \frac{1}{120}(1+m)(120 - 126m + 201m^2 - 216m^3 + 81m^4 + 250n - 300mn+ \\ 315m^2n - 135m^3n + 175n^2 - 180mn^2 + 90m^2n^2 + 50n^3 - 30mn^3 + 5n^4).\end{aligned} \tag{3.20}$$

As already mentioned, the two short exact sequences (3.12) imply two long exact sequences of the cohomology groups. If these long exact sequences contain enough zeros then one can hope to deduce the cohomology classes of the bundle V_M without a detailed study of the maps in these sequences. The Koszul sequence (3.17) does not simply imply one exact sequence in cohomology but instead gives rise to a spectral sequence or alternatively to the following three short exact sequences with their associated long cohomological sequences.

$$\begin{aligned}
&0 \to T \otimes \mathcal{O}(-1,-4) \to T \to T|_N \to 0 \\
&0 \to T \otimes \mathcal{O}(-2,-8) \to T \otimes \mathcal{O}(-1,-4) \to T \otimes \mathcal{O}(-1,-4)|_N \to 0 \\
&0 \to T \otimes \mathcal{O}(-1,-4)|_N \to T|_N \to T|_M \to 0,
\end{aligned} \tag{3.21}$$

where N denotes one of the two hypersurfaces, $M \subset N \subset A$.

For determining the generations and antigenerations, using (3.21) one first has to calculate the cohomology groups on M listed in Table 3.3.

	$\mathcal{O}\|_M$	$\mathcal{O}(1,0)\|_M$	$\mathcal{O}(0,1)\|_M$	$\mathcal{O}(0,2)\|_M$	$\mathcal{O}(1,5)\|_M$
h^0	1	1	5	15	131
h^1	0	0	0	0	0
h^2	0	0	0	0	0
h^3	1	0	0	0	0

Table 3.3: *Cohomology of line bundles on M .*

Then the first sequence in (3.12) implies the cohomology of $\mathcal{E}_M$ to be $h(\mathcal{E}_M) = (h^0, h^1, h^2, h^3) = (30, 0, 1, 0)$. From the second sequence one obtains $h(V_M) = (0, 101, 1, 0)$, thus the model has 101 generations in the **27** representation of E_6 and 1 antigeneration in the $\overline{\mathbf{27}}$ representation of E_6.

Carrying out the analogous computation for the tangent bundle of the base space, which is given by the two short exact sequences

$$\begin{aligned} &0 \to \mathcal{O} \oplus \mathcal{O}|_M \to \mathcal{O}(1,0) \oplus \mathcal{O}(0,1)^5 \oplus \mathcal{O}(1,3)|_M \to \mathcal{F}_M \to 0 \\ &0 \to T_M \to \mathcal{F}_M \to \mathcal{O}(1,4) \oplus \mathcal{O}(1,4)|_M \to 0, \end{aligned} \tag{3.22}$$

one gets $h(T_M) = (0, 86, 2, 0)$. Thus the base manifold has 86 complex deformations and 2 Kähler deformations. However, this is only part of the large radius moduli space of heterotic string compactifications. The remaining part are the bundle deformations parameterized by elements in $H^1(M, \mathrm{End(V_M)})$.

3.3. Bundle deformations

The bundle endomorphisms $\mathrm{End(V_M)}$ are by definition the traceless part of the bundle $V_M \otimes V_M^*$. Using $tr(V_M \otimes V_M^*) = \mathcal{O}|_M$ and that for stable bundles on a Calabi-Yau n-fold the lowest and highest cohomology groups H^0 and H^n vanish, one obtains the following relation between $h^q(M, \mathrm{End(V_M)})$ and $h^q(M, V \otimes V^*)$ [18]

$$h^q(M, V_M \otimes V_M^*) = \begin{cases} 1 & \text{for q=0,3} \\ h^q(M, \mathrm{End(V_M)}) & \text{for q=1,2.} \end{cases} \tag{3.23}$$

Furthermore, Serre duality implies $h^1(M, V_M \otimes V_M^*) = h^2(M, V \otimes V^*)$ which will provide a non-trivial check whether our calculation is correct.

To begin with, the exact sequence (3.12)(V) is dualized and tensored with V_M yielding the exact sequence

$$0 \to V_M \otimes \mathcal{O}(-1,-5)|_M \to V_M \otimes \mathcal{E}_M^* \to V_M \otimes V_M^* \to 0 \tag{3.24}$$

containing the desired bundle.

• $V_M \otimes \mathcal{O}(-1,-5)|_M$

To determine this bundle the following two exact sequences are used which are derived from (3.12).

$$\begin{aligned} 0 \to &V_M \otimes \mathcal{O}(-1,-5)|_M \to \mathcal{E}_M \otimes \mathcal{O}(-1,-5)|_M \to \mathcal{O}|_M \to 0 \\ 0 \to &\mathcal{O}(-1,-5)|_M \to \mathcal{O}(0,-5) \oplus \mathcal{O}(-1,-4)^3 \oplus \mathcal{O}(-1,-3)|_M \to \\ &\quad \mathcal{E}_M \otimes \mathcal{O}(-1,-5)|_M \to 0. \end{aligned} \tag{3.25}$$

Again one uses the Koszul sequence to determine the cohomology classes of all involved line bundles restricted to M and finally gets the results displayed in Table 3.4.

	$V_M \otimes \mathcal{O}(-1,-5)\|_M$	$\mathcal{E}_M \otimes \mathcal{O}(-1,-5)\|_M$	$\mathcal{O}\|_M$
h^0	0	0	1
h^1	1	0	0
h^2	0	0	0
h^3	248	249	1

Table 3.4: *determine* $V_M \otimes \mathcal{O}(-1,-5)|_M$

• $V_M \otimes \mathcal{E}_M^*$
The following sequence is used as the starting point to determine this bundle

$$0 \to V_M \otimes \mathcal{E}_M^* \to \mathcal{E}_M \otimes \mathcal{E}_M^* \to \mathcal{E}_M^* \otimes \mathcal{O}(1,5)|_M \to 0. \tag{3.26}$$

The cohomology of $\mathcal{E}_M^* \otimes \mathcal{O}(1,5)|_M$ is given by Serre duality as $h(\mathcal{E}_M^* \otimes \mathcal{O}(1,5)|_M) = (249,0,0,0)$.

• $\mathcal{E}_M \otimes \mathcal{E}_M^*$
Dualizing (3.12)(E) and tensoring with $\mathcal{E}_M$ gives the exact sequence

$$0 \to \mathcal{E}_M \otimes \mathcal{E}_M^* \to \mathcal{E}_M \otimes \big(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)^3 \oplus \mathcal{O}(0,-2)\big)\,|_M \to \mathcal{E}_M \to 0, \tag{3.27}$$

which allows one to determine $\mathcal{E}_M \otimes \mathcal{E}_M^*$ after having found the cohomology of all the $\mathcal{E}_M \otimes \mathcal{O}(m,n)|_M$. However, the latter can be computed analogously to $\mathcal{E}_M \otimes \mathcal{O}(-1,-5)|_M$ in the second sequence in (3.25). Tracing through all these sequences finally gives Table 3.5.

	$\mathcal{E}_M \otimes \mathcal{E}_M^*$	$\mathcal{E}_M \otimes \big(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)^3 \oplus \mathcal{O}(0,-2)\big)\,\|_M$	$\mathcal{E}_M$
h^0	11	$41 = 16 + 24 + 1$	30
h^1	0	0	0
h^2	0	0	1
h^3	11	$10 = 0 + 0 + 10$	0

Table 3.5: *determine* $\mathcal{E}_M \otimes \mathcal{E}_M^*$.

Here Serre duality was used to fix the possible additive offset to h^0 and h^1 to be zero. Going back to (3.26) one obtains that $V_M \otimes \mathcal{E}_M^*$ is fixed as displayed in Table 3.6.

	$V_M \otimes \mathcal{E}_M^*$	$\mathcal{E}_M \otimes \mathcal{E}_M^*$	$\mathcal{E}_M^* \otimes \mathcal{O}(1,5)\|_M$
h^0	x	11	249
h^1	$x+238$	0	0
h^2	0	0	0
h^3	11	11	0

Table 3.6: *determine* $V_M \otimes \mathcal{E}_M^*$.

The final step is to use (3.24) to determine $V_M \otimes V_M^*$ as shown in Table 3.7.

	$V_M \otimes \mathcal{O}(-1,-5)\|_M$	$V_M \otimes \mathcal{E}_M^*$	$V_M \otimes V_M^*$
h^0	0	x	1
h^1	1	$x+238$	238
h^2	0	0	238
h^3	248	11	1

Table 3.7: *determine* $V_M \otimes V_M^*$.

It is assumed here that the bundle is stable and due to (3.23) that $h^0 = h^3 = 1$. Thus, it is derived that the bundle has $h^1(M, \mathrm{End}(V_M)) = 238$ deformations. Adding up all the moduli at large radius one obtains $h^1(M,T)+h^2(M,T)+h^1(M,\mathrm{End}(V_M)) = 86+2+238 = 326$ which is the same number of moduli as for the quintic. Due to this nice non-trivial matching of the dimensions of the moduli spaces both at large and at small radius and the equality of the two models at their Landau-Ginzburg locus it is conjectured that the two moduli spaces are actually the same. A physicist doing experiments in four dimensions cannot decide whether the hidden six dimensional world is the quintic or the dual $(0,2)$ model.

It is clear that the mapping between the two sets of moduli must include exchanges between the three classes of moduli. For instance, at large radius of the dual $(0,2)$ model the vector bundle V_M is definitely different from the tangent bundle implying that the model is really $(0,2)$ and not $(2,2)$ supersymmetric. However, the quintic has a $(2,2)$ subset even for large radius. Thus, it cannot be that Kähler moduli are only mapped to Kähler moduli, instead Kähler moduli seem to be mapped to bundle moduli. This picture is also consistent with the fact that at the Landau-Ginzburg point there exist 25 twisted singlets [20]. For the quintic, only one of these singlets corresponds to the Kähler deformation, the other 24 are bundle deformations. For the dual $(0,2)$ model one expects two combinations of these 24 twisted singlets to constitute the new Kähler deformations.

It is tempting to speculate about a maximal $(0,2)$ model dual to the quintic, for which all 25 twisted singlets do correspond to some Kähler moduli.

Clearly, it would be very interesting to gain a better understanding of how the moduli spaces are actually mapped to each other. Similarly to mirror symmetry this would allow one to derive statements about some properties of the moduli space of one model by knowing it for the dual model. For instance, if the duality is correct, one can already learn that the moduli space of the $(0,2)$ model contains a 102 dimensional sublocus of $(2,2)$ world-sheet supersymmetry, which is absolutely not obvious by knowing the model only in the large radius phase. Using the results from mirror symmetry [12], one even knows the metric on the one dimensional subset of this $(2,2)$ locus which is mapped to the Kähler moduli space of the quintic.

An interesting observation one can make is that the base manifold of the $(0,2)$ model is related to the complete intersection manifold

$$\begin{array}{c} \mathbb{P}_4 \\ \mathbb{P}_1 \end{array} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.28}$$

by what is called a flip transition in [22]. This means that there exists a conifold transition from the quintic to the base manifold of the dual $(0,2)$ model. The duality proposed here has at first sight nothing to do with a singular transition and in the moment I do not know whether the appearance of the conifold transition here is only a coincidence or has really to teach us something about the non perturbative resolution of the conifold singularity in the heterotic string context.

4. The Sextic and its duals

In the last section one particular simple example was studied in very much detail revealing that in that case there cannot be a transition between different moduli spaces, instead we were led to the conjecture that the two moduli spaces are isomorphic. However, the quintic is special in the sense that at the Landau-Ginzburg locus the number of gauge singlets matches exactly the number of moduli at large radius. This fact excluded the possibility of a transition just from the very beginning. In this section another example will be studied, for which at the Landau-Ginzburg locus the number of singlets increases. Thus, this example is closer to the generic case than the quintic but fortunately it is still easy enough to be treated using the techniques from the last section.

4.1. The Sextic

The base manifold is a hypersurface of degree six in the weighted projective space $\mathbb{P}_{1,1,1,1,2}$ and the bundle is a deformation of the tangent bundle. At the Landau-Ginzburg point the model has $N_{27} = 103$ generations and $N_{\overline{27}} = 1$ antigenerations. Moreover, the spectrum contains $N_1 =$**340** gauge singlets, with 307, 27 and 6 arising from the $k = 1$, $k = 3$ and $k = 5$ twisted sector, respectively.

The number of moduli in the geometric phase is again the crucial unknown. Since one needs the cohomology classes of line bundles in the ambient variety as input for running the exact sequences, one has to worry about the $\mathbb{Z}_2$ singularity in the weighted projective space even though the hypersurface avoids this singularity.

Blowing up the singularity generates the charges of the fields defining the base manifold. These are shown in Table 4.1.

x_1	x_2	x_3	x_4	x_5	x_6	Γ_1
1	0	0	0	1	0	-2
2	1	1	1	0	1	-6

Table 4.1: *Charges for the base.*

However, for the tangent bundle one obtains $\chi(M) = -200$ which is not what one wants, but one can choose the bundle resolution in Table 4.2 which also agrees on the singular space $r_1 \to 0$ with the tangent bundle of the sextic.

λ_1	λ_2	λ_3	λ_4	λ_5	p
0	0	1	1	0	-2
1	1	1	1	2	-6

Table 4.2: *Charges for the bundle.*

Indeed, this model has $c_3(V_M) = -204$. Now one has to go all the way through the exact sequence calculation. One finds that there are $h^1(M, V_M) = 103$ generations and $h^2(M, V_M) = 1$ antigenerations. The base manifold has $h^1(M, T) = 102$ complex deformations and $h^2(M, T) = 2$ Kähler deformations. For the number of bundle deformations one obtains $h^1(M, \mathrm{End}(V_M)) = 234$, so that the total number of large radius moduli is **338**.

4.2. Dual model A

By introducing one new coordinate of weight $\omega = 4$ and exchanging some of the F_as with some of the hypersurface constraints G_j one obtains the singular model

$$V(1,1,2,2;6) \to \mathbb{P}_{1,1,1,1,2,4}[5,5], \tag{4.1}$$

which has the same Landau-Ginzburg potential as the $(0,2)$ sextic. The resolution of the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ singularity leads to the charges for the base manifold displayed in Table 4.3.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Γ_1	Γ_2
1	0	0	0	0	1	0	0	-1	-1
2	1	0	0	0	0	1	0	-2	-2
4	2	1	1	1	0	0	1	-5	-5

Table 4.3: *Charges for the base.*

The bundle resolution leads to the charges of the left moving fermions in Table 4.4.

λ_1	λ_2	λ_3	λ_4	λ_5	p
1	0	0	0	0	-1
0	1	1	0	0	-2
0	1	1	2	2	-6

Table 4.4: *Charges for the bundle.*

Using the Stanley-Reisner ideal

$$\mathrm{SR} = \{x_1\, x_6,\ x_2\, x_7,\ x_3\, x_4\, x_5\, x_8\}, \tag{4.2}$$

the intersection ring on the threefold is

$$8\eta_1^3 - 2\eta_1\eta_2^2 + 8\eta_2^3 - 2\eta_1^2\eta_3 + \eta_1\eta_2\eta_3 - 4\eta_2^2\eta_3 + 2\eta_2\eta_3^2 + 2\eta_3^3. \tag{4.3}$$

The third Chern class of the bundle and the Euler characteristic of the base manifold are $c_3(V_M) = -204$ and $\chi(M) = -176$, respectively. The exact sequence calculation reveals that the model has $h^1(M, V_M) = 103$ generations, $h^2(M, V_M) = 1$ antigenerations, $h^1(M, T_M) = 91$ complex deformations and $h^2(M, T_M) = 3$ Kähler deformations. After another lengthy calculation one gets $h^1(M, \mathrm{End}(V_M)) = 244$ bundle deformations, so that the total number of moduli comes out to be **338**, the same as for the sextic. Thus, even though in this case the Landau-Ginzburg model has two more singlets than the geometric models, the number of moduli in the geometric phases agree completely! This surprising coincidence is considered as a sign that also in this case there is no transition between different moduli spaces but rather an isomorphy between them.

4.3. Dual model B

The duality between the sextic and model A can even be extended to a triality. The singular model

$$V(1,1,1,3;6) \to \mathbb{P}_{1,1,1,1,2,3}[5,4] \tag{4.4}$$

agrees at its Landau-Ginzburg locus with the sextic and dual model A. The resolution of the base and the bundle results in the assignment of charges shown in Table 4.5 and Table 4.6, respectively.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Γ_1	Γ_2
1	0	0	0	0	1	0	0	-1	-1
2	1	0	0	0	0	1	0	-2	-2
3	2	1	1	1	0	0	1	-5	-4

Table 4.5: *Charges for the base.*

λ_1	λ_2	λ_3	λ_4	λ_5	p
1	0	0	0	0	-1
0	1	1	0	0	-2
0	1	1	1	3	-6

Table 4.6: *Charges for the bundle.*

Here one obtains $c_3(V_M) = -204$ and $\chi(M) = -160$. The more refined exact sequence calculation gives $h^1(M,V_M) = 103$ generations, $h^2(M,V_M) = 1$ antigenerations, $h^1(M,T_M) = 83$ complex deformations and $h^2(M,T_M) = 3$ Kähler deformations. Furthermore, the model has $h^1(M,\mathrm{End}(V_M)) = 252$ bundle deformations so that the total number of moduli is again **338**.

5. Conclusion

It is clear that many more dual models could be studied in a similar way. For instance, applying an exchange of G_j and F_a constraints to the model $\mathbb{P}_{1,1,1,2,2}[7]$, one obtains the following three singular dual candidates

$$\begin{aligned} &V(1,1,1,4;7) \to \mathbb{P}_{1,1,1,2,2,3}[5,5] \\ &V(1,1,2,3;7) \to \mathbb{P}_{1,1,1,2,2,4}[6,5] \\ &V(1,2,2,2;7) \to \mathbb{P}_{1,1,1,2,2,5}[6,6], \end{aligned} \tag{5.1}$$

which would lead to four models with isomorphic moduli spaces.

Throughout this paper, we have restricted ourselves to coherent sheaves, which actually are vector bundles. As nicely shown in [7], perturbative string theory can very well live with some mild singularities in the bundle leading to reflexive or torsion free sheaves. It would be interesting whether the proposed duality extends to this more general case.

It might be possible to find a dual pair for which both base manifolds are elliptically fibered. In this case the proposed duality should have an analogue in the F-theory dual picture. It might also be that this duality of $(0,2)$ compactifications is not only limited to models which allow a Landau-Ginzburg description, but extended to more general $(0,2)$ models.

To summarize, by performing an exact cohomology calculation, it was shown that not only at small radius but also at large radius the dimensions of the moduli spaces of the quintic and its dual $(0,2)$ model agree. This matching generalizes even to the case when at the Landau-Ginzburg locus additional singlets appear. This result was viewed as strong indication that the moduli spaces of the models involved are in fact isomorphic and one is actually dealing with a perturbative target space duality.

Acknowledgments

It is a pleasure to thank Michael Flohr, Peter Mayr and Andreas Wißkirchen for discussion. This work is supported by NSF grant PHY–9513835.

References

[1] T. Banks, L.J. Dixon, D. Friedan and E. Martinec, *Phenomenology and conformal field theory, or Can string theory predict the weak mixing angle?*, Nucl. Phys. **B299** (1988) 613

[2] E. Witten, *New issues in manifolds of* $SU(3)$ *holonomy*, Nucl. Phys. **B268** (1986) 79;
M. Dine, N. Seiberg, X. Wen and E. Witten, *Nonperturbative effects on the string world sheet I,II*, Nucl. Phys. **B278** (1986) 769; *ibid.* **B289** (1987) 319;
J. Distler, *Resurrecting* $(2,0)$ *compactifications*, Phys. Lett. **B188** (1987) 431;
J. Distler and B. R. Greene, *Aspects of* $(2,0)$ *string compactifications*, Nucl. Phys. **B304** (1988) 1

[3] J. Distler and S. Kachru, $(0,2)$ *Landau–Ginzburg theory*, Nucl. Phys. **B413** (1994) 213, hep–th/9309110;
J. Distler, *Notes on* $(0,2)$ *superconformal field theories*, in Trieste HEP Cosmology 1994,hep–th/9502012

[4] J. Distler and S. Kachru, *Duality of* $(0,2)$ *String Vacua*, Nucl. Phys. **B442** (1995) 64, hep-th/9501111

[5] J. Distler and S. Kachru, *Singlet couplings and* $(0,2)$ *models*, Nucl. Phys. **B430** (1994) 13, hep–th/9406090;
E. Silverstein and E. Witten, *Criteria for conformal invariance of* $(0,2)$ *models*, Nucl. Phys. **B444** (1995) 161, hep–th/9503212

[6] R. Blumenhagen and A. Wißkirchen, *Exactly solvable* $(0,2)$ *supersymmetric string vacua with GUT gauge groups*, Nucl. Phys. **B454** (1995) 561, hep–th/9506104;
R. Blumenhagen, R. Schimmrigk and A. Wißkirchen, *The* $(0,2)$ *exactly solvable structure of chiral rings, Landau–Ginzburg theories and Calabi–Yau manifolds*, Nucl. Phys. **B461** (1996) 460, hep–th/9510055 ;
R. Blumenhagen and A. Wißkirchen, *Exploring the moduli space of* $(0,2)$ *strings*, Nucl. Phys. **B475** (1996) 225, hep-th/9604140

[7] J. Distler, B. R. Greene and D. R. Morrison, *Resolving singularities in* $(0,2)$ *models*, Nucl. Phys. **B481** (1996) 289, hep–th/9605222;
J. Distler, *private communication*

[8] T.-M. Chiang, J. Distler, B. R. Greene, *Some Features of* $(0,2)$ *Moduli Space*, Nucl. Phys. **B496** (1997) 590, hep-th/9702030

[9] P. Berglund, C. V. Johnson, S. Kachru and P. Zaugg, *Heterotic coset models and* $(0,2)$ *string vacua*, Nucl. Phys. **B460** (1996) 252, hep-th/9509170;
M. Kreuzer and M. Nikbakht-Tehrani, $(0,2)$ *string compactifications*, Talk at the International Symposium on the Theory of Elementary Particles, Buckow, August 27-31, hep-th/9611130

[10] R. Blumenhagen, R. Schimmrigk and A. Wißkirchen, $(0,2)$ *mirror symmetry*, Nucl. Phys. **B486** (1997) 598, hep–th/9609167 ;
R. Blumenhagen and S. Sethi, *On orbifolds of* $(0,2)$ *models*, Nucl. Phys. **B491** (1997) 263, hep–th/9611172 ;
R. Blumenhagen and M. Flohr, *Aspects of* $(0,2)$ *orbifolds and mirror symmetry*, IASSNS-HEP-97/13, hep–th/9702199

[11] B. R. Greene and R. Plesser, *Duality in Calabi–Yau moduli space*, Nucl. Phys. **B338** (1990) 15;
P. Candelas, M. Lynker and R. Schimmrigk, *Calabi–Yau manifolds in weighted* $\mathbb{P}_4$, Nucl. Phys. **B341** (1990) 383

[12] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A pair of calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. **B359** (1991) 21

[13] P. S. Aspinwall, B. R. Greene and D. R. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, Nucl. Phys. **B416** (1994) 414, hep–th/9309097;
V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3** (1994) 493, alg–geom/9310003

[14] S. Kachru and C. Vafa, *Exact results for* $N=2$ *compactifications of heterotic strings*, Nucl. Phys. **B450** (1995) 69, hep–th/9505105

[15] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, *Nonperturbative results on the point particle limit of* $N=2$ *heterotic string compactifications*, Nucl. Phys. **B459** (1996) 53, hep-th/9508155

[16] S. Sethi, C. Vafa, E. Witten, *Constraints on low-dimensional string compactifications*, Nucl. Phys. **B480** (1996) 213-224, hep-th/9606122 ;
I. Brunner, M. Lynker and R. Schimmrigk, *F-theory on Calabi-Yau fourfolds*, Phys. Lett. **B387** (1996) 750, hep-th/9606148 ;
P. Mayr, *Mirror symmetry,* $N=1$ *superpotentials and tensionless strings on Calabi-Yau four-folds*, Nucl. Phys. **B494** (1997) 489, hep-th/9610162 ;
R. Friedman, J. Morgan and E. Witten, *Vector bundle and F-theory*, IASSNS-HEP-97-3, hep-th/9701162 ;
M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *Four dimensional compactification of F-theory*, HUTP-96-A054, hep-th/9701165

[17] A. Strominger, *Massless black holes and conifolds in string theory*, Nucl. Phys. **B451** (1995) 96, hep–th/9504090;
B. R. Greene, D. R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua*, Nucl. Phys. **B451** (1995) 109, hep–th/9504145

[18] J. Distler, B. R. Greene, K. Kirklin and P. Miron, *Calculating endomorphism valued cohomology: Singlet spectrum in superstring models*, Commun. Math. Phys. **122** (1989) 117;

M. G. Eastwood and T. Hübsch, *Endomorphism valued cohomology and gauge neutral matter*, Commun. Math. Phys. **132** (1990) 383;

T. Hübsch, *Calabi-Yau manifolds*, World Scientific 1992;

P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Willey & Sons 1978

[19] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159, hep–th/9301042

[20] S. Kachru and E. Witten, *Computing the complete massless spectrum of a Landau–Ginzburg orbifold*, Nucl. Phys. **B407** (1993) 637, hep–th/9307038

[21] S. Katz and S. A. Strømme, *Schubert: a maple package for intersection theory*;
J. DeLoera, *Puntos*

[22] P. Candelas, P. S. Green and T. Hübsch, *Finite distances between distinct Calabi–Yau vacua: (other worlds are just around the corner)*, Phys. Rev. Lett. **62** (1989) 1956; *Rolling among Calabi–Yau vacua*, Nucl. Phys. **B330** (1990) 49